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Comparative Analysis of Time Steps Distribution in Runge-Kutta Algorithms

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Abstract— This study utilized combination of phase plots,time steps distribution and adaptive time steps Runge-Kutta and fifth order algorithms to investigate a harmonically Duffing oscillator.The object is to visually compare fourth and fifth order Runge-Kutta algorithms performance as tools for seeking the chaotic solutions of a harmonically excited Duffing oscillator.Though fifth order algorithms favours higher time steps and as such faster to execute than fourth order for all studied cases.The reliability of results obtained w ith fourth order worth its higher recorded total computation time steps period.

Keywords—Algorithms, Chaotic Solutions, Duffing Oscillator, Harmonically Excited, Phase Plots, Runge-Kutta and Time Steps

1 Introduction

Extensive literature study shows that numerical technique is very important in obtaining solutions of differential equations of nonlinear systems.The most common universally accepted numerical techniques are Backward differential formulae, Runge-Kutta and Adams-Bashforth-Moulton. According to Julyan and Oreste in 1992, Runge-Kutta family of algorithms remain the most popular and used methods for integration. In numerical analysis, the Runge-Kutta methods can be classified as important family of implicit and explicit iterative methods for the approximations of solutions of ordinary differential equations. Historically, the Runge-Kutta techniques were developed by the German mathematicians C.Runge and M.W. Kutta. The combination of the two names formed the basis of nomenclature of the method known as Runge-Kutta. The relevance of Runge-Kutta algorithms in finding solutions to problems in nonlinear dynamics cannot be overemphasized. Quite a number of research efforts have been made in the numerical solutions of nonlinear dynamic problems. It is usual when investigating the dynamics of a continuous-time system described by an ordinary differential equation to first investigate in order to obtain trajectories. Julyan and Oreste (1992) were able to elucidate the dynamics of the most commonly used family of numerical integration schemes (Runge-Kutta methods). The study of the authors showed that Runge-Kutta integration should be applied to nonlinear systems with knowledge of caveats involved. Detailed explanation was provided for the interaction between stiffness and chaos.The findings of this research revealed that explicit Runge-Kutta schemes should not be used for stiff problems mainly because of their inefficiency. According to the authors, the best alternative method is to employ Backward differentiation formulae methods or possibily implicit Runge-Kutta methods.

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The conclusions drawn from the paper elucidated the fact that dynamics is not only interested in problems with fixed point solutions, but also in periodic and chaotic behaviour.

The application of bifurcation diagrams in the chaotic study of nonlinear electrical circuits has been demonstrated (Ajide and Salau, 2011). The relevant second order differential equations were solved for ranges of appropriate parameters using Runge-Kutta method.The solutions obtained from this method were employed to produce bifurcation diagrams. This paper showed that bifurcation diagram is a useful tool for exploring dynamics of nonlinear resonant circuit over a range of control parameters. Ponalagusamy 2009 research paper focused on providing numerical solutions for system of second order robot arm problem using the Runge-Kutta sixth order algorithm. The precised solution of the system of equations representing the arm model of a robot has been compared with the corresponding approximate solutions at different intervals. The results and comparison showed that the efficiency of numerical integration algorithm based on the absolute error between the exact and approximate solutions. The implication of this finding is that STWS algorithm is not based on Taylor's series and it is an A-stable method. The dynamics of a torsional system with harmonically varying drying friction torque was investigated by Duan and Singh (2008). Nonlinear dynamics of a single degree of freedom torsional system with dry friction is chosen as a case study. Nonlinear system with a periodically varying normal load was first formulated. This is followed by re-formulation of a multi-term harmonic balance method (MHBM). The reason for this is to directly solve the nonlinear time-varying problem in frequency domain. The feasibility of MHBM is demonstrated with a periodically varying friction and its accuracy is validated by numerical integration using fourth order Runge-Kutta scheme. The set of explicit third order new improved Runge-Kutta (NIRK) method that just employed two function evaluations per step has been developed (Mohamed et al, 2011). Due to lower number of function evaluations, the scheme proposed herein has a lower computational cost than the classical third order Runge-Kutta method while maintaining the same order of local accuracy. Bernardo and Chi-Wang (2011) carried out a critical review on the development of Ruge-Kutta discontinuous Galerkin (RKDG) methods for nonlinear convection dominated problems. The

authors combined a special class of Runge-Kutta time discretizations that allows the method to be nonlinearly stable regardless of its accuracy with a finite element space discretization by discontinuous approximations that incorporates the idea of numerical fluxes and slope limiters coined during the remarkable development of high resolution finite difference and finite volume schemes. This review revealed that RKDG methods are stable, high-order accurate and highly parallelizable schemes that can easily handle complicated geometries and boundary conditions.The review showed its immense applications in Navier-Stokes equations and Hamilton-Jacobian equations. This study no doubt has brought a relief in computational fluid dynamics.This technique has been mostly employed in analyzing Duffing oscillator dynamics.The Duffing oscillator has been described as a set of two simple coupled ordinary differential equations to solve . Runge-Kutta method has been extensively used for numerical solutions of Duffing oscillator dynamics. Salau and Ajide (2011) investigated the dynamical behaviour of a Duffing oscillator using bifurcation diagrams. The authors employed fourth order Runge-Kutta method in solving relevant second order differential equations. While the bifurcation diagrams obtained revealed the dynamics of the Duffing oscillator, it also shows that the dynamics depend strongly on initial conditions. Salau and Oke (2010) showed how Duffing equation can be applied in predicting the emission characteristics of sawdust particles. The paper explains the modeling of sawdust particle motion as a two dimensional transformation system of continous time series. The authors employed Runge-Kutta algorithm in providing solution to Duffing's model equation for the sawdust particles. The solution was based on displacement and velocity perspective. The findings of the authors showed a high profile feasibility of modeling sawdust dynamics as emissions from band saws. The conclusion drawn from this work is that the finding no doubt provides advancement in the knowledge of sawdust emission studies.

Despite this wide application of Runge-Kutta method as a numerical tool in nonlinear dynamics, there is no iota of doubt that a research gap exists. Available literature shows that a research which compares the performance of different order (Second, Third, Fifth, Sixth e.t.c.) of Runge-Kutta has not been carried out. The objective of this paper is to visually compare fourth and fifth order Runge-Kutta algorithms performance as tools for seeking the chaotic solutions of a harmonically excited Duffing oscillator.

2 METHODOLOGY

2.1 Duffing Oscillator

The studied normalized governing equation for the dynamic behaviour of harmonically excited Duffing system is given by equation (1)

$$\ddot{x} + \gamma\dot{x}\left(1 - x^2\right) = P_0\left(Sin\,\omega t\right) \qquad (1)$$

In equation (1); $x, \dot{x}, \ddot{x}$ represents respectively displacement, velocity and acceleration of the Duffing oscillator about a set datum. The damping coefficient is γ . Amplitude strength of harmonic excitation, excitation frequency and time are respectively P_0, ω and t. Francis (1987), Dowell (1988) and Narayanan and Jayaraman (1989b) proposed that the combination of $\gamma = 0.168$, $P_0 =$ 0.21 and $\omega = 1.0$ or $\gamma = 0.0168$, $P_0 =$ 0.09 and $\omega = 1.0$ parameters leads to chaotic behaviour of harmonically excited Duffing oscillator.This study utilized adaptive time steps Runge-Kutta algorithms to investigate equation (1) over one hundred and fifty excitation starting with a time step of ($\Delta t =$ Excitation Period/1000). The phase plot was made with the stable solutions from the last fifty (50) excitation period calculations.

2.2 Time Step Selection

Steven and Raymond (2006) argued that employing a constant step size to seek solutions of ordinary differential equations of some dynamical systems that exhibits an abrupt change could pose serious limitation.In such engineering problems (chaotic dynamics) of interest,the choice of adaptive time step size becomes inevitable. The formula used for increasing and decreasing the time step (Δt) in this study is given by (2) and (3) respectively.The tolerance (ε_t) was fixed at 10^{-6} for all computation steps while the error (ε) compares predicted results taking two half-steps with taking a full step called module-1. Similarly module-2 compares predicted results taking three one third with taking a full step. Equation (2) is used when $\varepsilon < \varepsilon_t$ and equation (3) is used when $\varepsilon > \varepsilon_t$.

$$\Delta t = \Delta t(0.95)(\varepsilon_t/\varepsilon)^{1/4} \qquad (2)$$

$$\Delta t = \Delta t(0.95)(\varepsilon_t/\varepsilon)^{1/5} \qquad (3)$$

2.3 Parameter Details of Studied Cases

Three different cases were studied using the details given in table 1 in conjunction with governing equation (1).Common parameters to all cases includes displacement $(x = 1.0)$, Zero initial velocity $(\ddot{x})$ and excitation frequency $(\omega = 1)$.

Table 1 : Combined Parameters for Cases

Cases	Damping Coefficient (γ)	Excitation Amplitude (P_0)
Case -1	0.1680	0.21
Case-2	0.0168	0.09
Case-3	0.0168	0.21

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3 RESULTS AND DISCUSSIONS

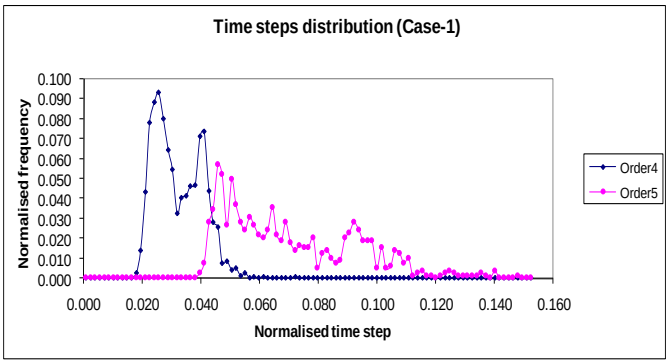

Fig.1: Comparative Time Steps Distribution (Case-1) of fouth and fifth order Runge-Kutta Algorithms

Fig.1 refers; the time steps distribution range is shorter for the fourth order algorithms and longer for the fifth order algorithms.The fourth order algorithms is less tolerant of higher computational time steps than fifth order algorithm.The distributins for the fourth and fifth order algorithms peaked at 0.026 and 0.026 excitation periods respectively.

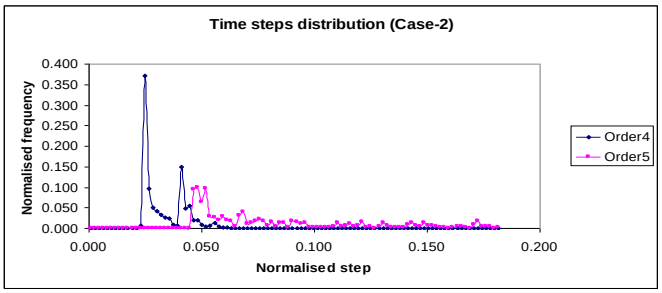

Fig.2: Comparative Time Steps Distribution (Case-2) of fourth and fifth Order Runge-Kutta Algorithms

Fig.2 can be interpreted qualitatively as figure 1. However the frequency intensities differ drastically. The distributions for the fourth and fifth order algorithms peaked at 0.025 and 0.048 excitation periods respectively.

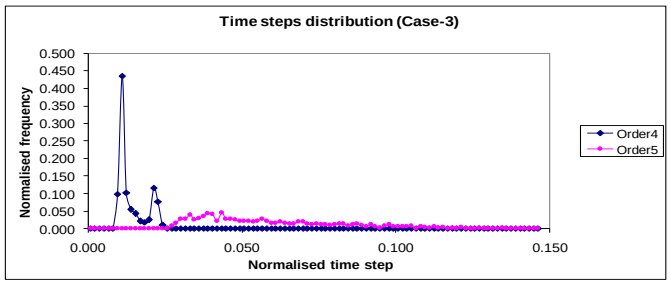

Fig.3: Comparative Time Steps Distribution (Case-3) of fourth and fifth Order Runge-Kutta Algorithms

Fig.3 can be interpreted qualitatively as figures 1 and 2. However, the frequency intensities differ and the distributions for the fourth and fifth order algorithms peaked at 0.011 and 0.043 periods respectively.

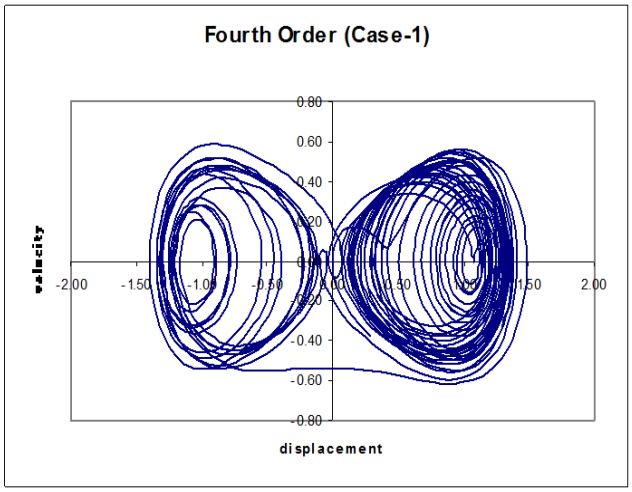

Fig.4a: Phase Plot Obtained for fourth Order (Case-1)

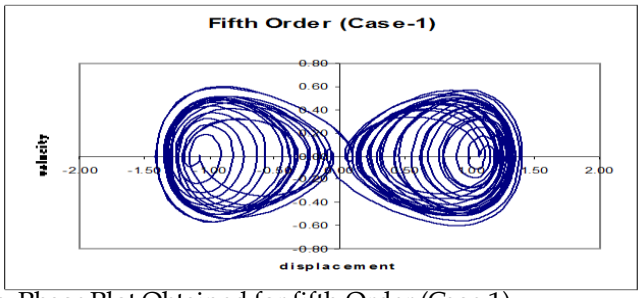

Fig.4b: Phase Plot Obtained for fifth Order (Case-1)

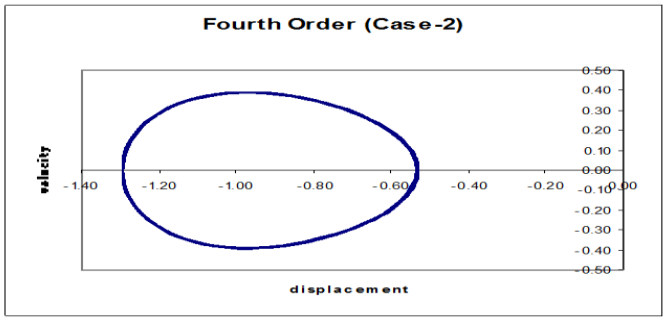

Fig.4c: Phase Plot Obtained for fourth Order (Case-2)

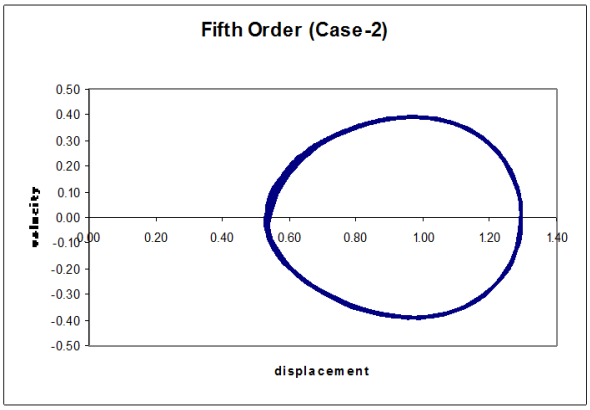

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Fig.4d: Phase Plot Obtained for fifth Order (Case-2)

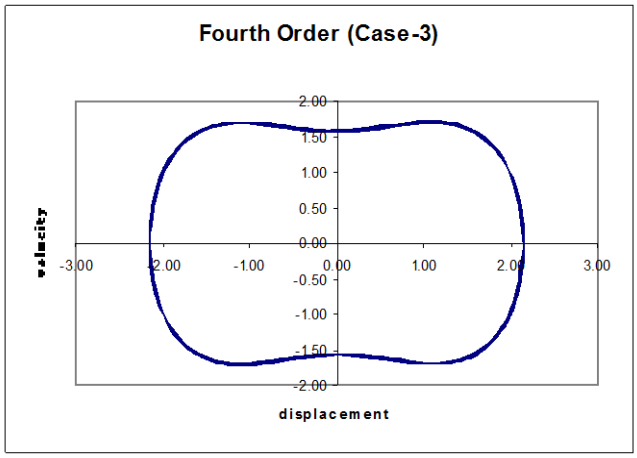

Fig.4e: Phase Plot Obtained for fourth Order (Case-3)

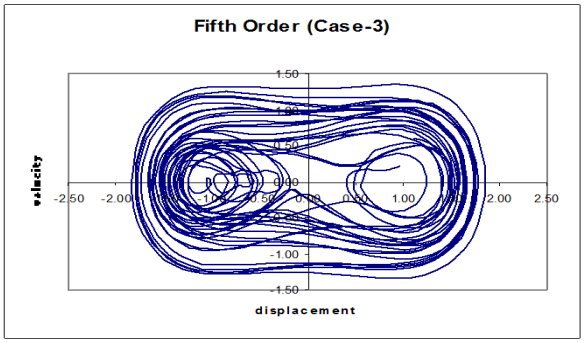

Fig.4f: Phase Plot Obtained for fifth Order (Case-3)

Fig.4 (a-f) shows the comparison of the phase plots obtained using Runge-Kutta fourth and fifth orders (module-1). Fig.4 (a-f) refers; the phase plots are only similar but not exact for case-1 and case-2 only. A closer observation of the phase plot for case-2 shows that solutions obtained by fourth order algorithm are bounded to the negative side of the displacement while the solutions obtained by fifth order algorithm are bounded to the positive side of the displacement.The phase plot in figure 4 (a) compare very well with phase plot obtained by Dowell (1988) . In addition,interpretations of table 2 strongly support higher consistency and reliability of fourth order algorithm results than its fifth order counterpart.Overall comparative assessment of the phase plots in conjunction with time steps distribution suggest fourth order algorithm results as more reliable than fifth order at the expense of more computation steps period(see table 3).Table 3 shows that adaptive fourth order can be twenty five(25) time fast to execute comparing with its equivalent constant time steps (See case-1 and case-2). Similarly, adaptive fifth order can be fifty (50) time fast to execute comparing with its equivalent time steps (See all cases). Table 3 further shows that adaptive fifth order can be four times fast to compute as its counterpart fourth order as recorded in case-3. However, reliability of computed results may be doubtful. The ratio of total number of steps taking to seek steady solutions by fourth and fifth order algorithms is module independent.

Table 2a: Corresponding Phase Plots Referring to fig.4 (a, c, e) for Fourth Order Algorithm

Cases	Constant Time Steps	Adjustable Time Steps(Module-1)	Adjustable Time Steps(Module-2)
Case-1	A	A	None
Case-2	D	C	Rough C
Case-3	E	E	Rough E

Table 2b: Corresponding Phase Plots Referring to fig. 2(b, d, f) for Fifth Order Algorithm

Cases	Constant Time Steps	Adjustable Time Steps(Module-1)	Adjustable Time Steps(Module-2)
Case-1	None	B	Fair A
Case-2	Looks Closer to B	D	C
Case-3	E	F	F

Note : A,B,C,D,E,F is the same as fig.4(a),(b),(c),(d),(e) and (f) respectively.

Table 3a: Total Number of Variable Steps Taken to Obtain the Steady Solutions within Studied 50 Excitation Periods (Fourth Order Runge-Kutta)

Cases	Constant Time Steps	Adaptive Time Steps(Module-1)	Adaptive Time Steps(Module-2)
Case-1	50000	1598	1588
Case-2	50000	1598	1594
Case-3	50000	3667	3704

Table 3b: Total Number of Variable Steps Taken to Obtain the Steady Solutions within Studied 50 Excitation Periods (Fourth Order Runge-Kutta)

Cases	Constant Time Steps	Adaptive Time Steps(Module-1)	Adaptive Time Steps(Module-2)
Case-1	50000	785	764
Case-2	50000	788	792
Case-3	50000	962	959

Table 3c: Ratio of Total Number of Steps in Fourth Order to Fifth Order

Module-1	Module-2
2	2
2	2
4	4

4 CONCLUSIONS

This study has visually illustrated the performance of two Runge-Kutta algorithms to seek the chaotic steady solutions of harmonically excited Duffing oscillator. The study has shown that Runge-Kutta fifth order can be four time fast to execute comparing with the corresponding fourth order but at the expense of reliability of the computed results.

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